Nonlinear Alfvén waves in magnetized plasmas with heavy impurities or dust

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(Received 24 April 1996; revised manuscript received 15 August 1996)

Nonlinear electromagnetic wave propagation in a magnetized plasma with heavy impurities, in particular, dust grains, at frequencies below and of the order of the ion-cyclotron frequency is considered. The presence of immobile dust grains leads to a considerable change of the dispersion characteristics for Alfvén waves propagating along the uniform magnetic field, especially for small wave numbers when two modes with a relatively large frequency gap appear. We discuss the consequences of this dispersion for the nonlinear properties of the waves. [S1063-651X(96)07712-4]

PACS number(s): 52.35.Bj, 52.35.Mw, 52.35.Sb, 52.25.Vy

I. INTRODUCTION

The study of Alfvén waves is important for magnetized space plasmas. When the amplitudes of the waves are large, nonlinear effects become essential in their propagation. It has been demonstrated that in plasmas with uniform density and background magnetic field, a balance between nonlinearity and dispersion can be established for the circularly polarized waves propagating along the external magnetic field, leading to the formation of solitary wave structures [1–5]. The evolution of the wave magnetic field is governed by the derivative nonlinear Schrödinger equation (DNLS) which is one of the canonical equations solved by the inverse scattering transform [6–8].

Many space plasmas contain charged dust and the presence of such a heavy (compared with the masses of ions) component can strongly affect the collective processes in the plasma. Thus recently there has been much interest in the investigation of a dusty plasma, i.e., a multicomponent low-temperature ionized gas containing electrons, ions, and charged dust particulates of a micrometer size. The charging of the dust is due to various effects such as plasma currents, photoemission, etc. Here, we note that numerous applications of dusty plasma studies include not only plasmas in space and the earth's environment [9,10], but also laboratory plasmas, expecially those for the manufacture of microelectronic components [11,12].

The presence of dust particles changes many plasma processes. One of the most important effects is the collection of electrons and ions from the background plasma by the charged grains. This affects the equilibrium state when the electron and ion charge balance necessarily includes the charge and density of the particulates

$$-en_e+en_i-Z_d e n_d=0. \qquad (1)$$

Here $n_{e,i,d}$ is the concentration of plasma electrons (with the charge $-e$), ions (for simplicity, we consider singly charged ions), and dust grains, respectively. For many dusty plasmas, the grain charge is negative (i.e., $Z_d>0$) and large ($Z_d\sim 10^2-10^3$), so that an appreciable proportion of the negative charge in the plasma may reside on the dust particles. In that case the dispersion properties of Alfvén in the plasma are strongly modified by the dust, because the ion Hall current is not compensated by the electron Hall current at low frequencies as well as at frequencies comparable with the ion-cyclotron frequency, so that ion-cyclotron effects extend to frequencies much less than the ion-cyclotron frequency [14]. There is an analogy to the case of electromagnetic waves in plasmas in solids with unequal electron and hole numbers [15]. The effect of the charge imbalance on the dispersion relation of linear Alfvén and magnetoacoustic waves in a dusty plasma has been investigated previously by a number of authors [14,16]. Reference [16] focused on the very low frequency (much less than the ion-cyclotron) waves that are affected by the motion of the dust grains themselves, while [14] considered the circularly polarized electromagnetic waves propagating parallel to the magnetic field in a plasma with static dust grains, in particular, the case of frequency much less than the ion-cyclotron frequency. Low-frequency dusty plasma modes taking into account dust grain motions and plasma temperature effects have also been studied in recent paper [17]. Furthermore, nonlinear electromagnetic modes in dusty plasmas at very low frequencies with dust grain motion and dust charge fluctuations have been considered in [18–20]. It was also demonstrated that under certain conditions a whistlerlike instability in a dusty plasma with streaming electrons and ions may develop in the presence of the charge fluctuations [21].

In dust-free plasmas, the low-frequency electromagnetic waves have the usual shear and compressional Alfvén wave properties, while in the presence of the dust grains the waves are better described as circularly polarized whistler or helicon waves extending to low frequencies [14,22]. The waves at frequencies comparable to the ion-cyclotron frequency in dusty plasmas are also circularly polarized modes, as in the dust-free case. Recently, it has been demonstrated [22] that the Alfvén resonance process is strongly modified in the presence of dust grains because of the imbalance of electron and ion charges.

In this paper, we study nonlinear effects in the electromagnetic wave propagation parallel to the uniform back-

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ground magnetic field in a magnetized plasma with cold electrons and stationary dust grains. The wave frequencies are less than or of the order of the ion-cyclotron frequency. We neglect the influence of the dust motions (see, e.g., [17–21]) as well as the dust charge fluctuation effects [18–21,23]. We show that since the linear dispersion of the waves at frequencies of order and less than the ion-cyclotron frequency is radically altered when a proportion of the negative charge resides on the dust grains, the nonlinear properties of the modes are quite different from the well-known dust-free case. For small wave numbers, two circularly polarized small amplitude modes with relatively large frequency gap (depending on the electron-ion charge imbalance) occur, and the nonlinear properties of these two modes are quite distinct. For the higher-frequency mode, the character of the dispersion and nonlinearity leads to the mixed DNLS-NLS (nonlinear Schrödinger) equation for the field envelope evolution in the case of a constant profile traveling wave. For the low-frequency mode, the nonlinear effects in the presence of the immobile dust are shown to be of lower order, and the dust grain dynamics as, e.g., in Refs. [17–21], should be included in consideration of this mode.

II. WAVE EQUATIONS

We invoke the standard two-fluid magnetohydrodynamic (MHD) model (see, e.g., [24]), which includes the fluid momentum equations for the plasma (singly charged cold) ions and (inertialess) electrons, the ion continuity equation, as well as Maxwell's equations ignoring the displacement current. The background magnetic field $\mathbf{B}_0$ is in the z direction, and the dust grains have infinite masses (we thus exclude dust dynamics from consideration). Furthermore, we introduce the parameter $\delta = n_e/n_i$ which measures the charge imbalance in the plasma. The total system is supposed to be neutral with the remainder of the charge residing on the dust particles according to Eq. (1).

The starting equations for the ion and electron velocities $\mathbf{v}_i$ and $\mathbf{v}_e$, the wave electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ (the latter also includes the background field $\mathbf{B}_0$), and the ion and electron densities n_i and n_e are given by

$$m_i n_i \frac{d\mathbf{v}_i}{dt} = e n_i (\mathbf{E} + \mathbf{v}_i \times \mathbf{B}), \tag{2}$$

$$0 = -e n_e (\mathbf{E} + \mathbf{v}_e \times \mathbf{B}), \tag{3}$$

$$\frac{\partial n_i}{\partial t} + \boldsymbol{\nabla}(n_i \mathbf{v}_i) = 0, \tag{4}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{5}$$

and

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 e (n_i \mathbf{v}_i - n_e \mathbf{v}_e). \tag{6}$$

Here, m_i is the ion mass. We have neglected the electron inertia in Eq. (3), which is equivalent to assuming that the wave frequencies of interest are much less than the electron cyclotron frequency Ω_e. We have also neglected the displacement current in Eq. (6).

By eliminating the electron variables from Eqs. (2)–(6), and using Faraday's law, we find the following nonlinear MHD (with Hall term) system of equations, modified due to the charge imbalance

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}(\rho \mathbf{v}) = 0, \tag{7}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla}(\mathbf{v} \times \mathbf{B}) - \frac{1}{\Omega_i} \boldsymbol{\nabla} \times \frac{d\mathbf{v}}{dt}, \tag{8}$$

and

$$\delta \frac{d\mathbf{v}}{dt} = -(1-\delta)\Omega_i \mathbf{v} \times \mathbf{B} + \frac{v_A^2}{\rho}(\boldsymbol{\nabla} \times \mathbf{B})\mathbf{B}, \tag{9}$$

where for convenience we have introduced the dimensionless magnetic field amplitude normalized as $\mathbf{B} = \mathbf{B}/B_0$, where $B_0 = |\mathbf{B}_0|$, and we have omitted the subscript i for the ion velocity. In Eqs. (7)–(9), $\rho = m_i n_i/\rho_0$ is the normalized density of the ion component of the plasma, where $\rho_0 = m_i n_{i0}$ and n_{i0} corresponds to the ion concentration in the state of equilibrium, $\Omega_i = eB_0/m_i$ is the ion gyrofrequency, $v_A = (B_0^2/\mu_0 \rho_0)^{1/2}$ is the Alfvén velocity based on the equilibrium ion density. The charge imbalance in the impurity-containing plasma introduces the first term on the right hand side of Eq. (9) and modifies the last one there. Equations (7)–(9) reduce, in the case of a dust-free plasma ($\delta = 1$ assuming quasineutrality is maintained), to the equations used in [1–5] in a treatment of nonlinear Alfvén waves.

We note that Eqs. (7)–(9) are only valid if the frequency is much less than the electron cyclotron frequency and the displacement current can be neglected. We show in Sec. III that the first condition requires that

$$\delta \gg \frac{m_e}{m_i}. \tag{10}$$

The electron current in Eq. (6) is much greater than the displacement current if

$$\mu_0 e n_e v_e \gg \frac{\omega}{c^2} E, \tag{11}$$

which may be written, using $v_e \approx E/B_0$,

$$\delta \gg \frac{\omega}{\Omega_i} \frac{v_A^2}{c^2}, \tag{12}$$

while the ion current is much greater than the displacement current if

$$\frac{\Omega_i}{\omega} \gg \frac{v_A^2}{c^2}. \tag{13}$$

Since we are mostly interested in the frequency range of order or less than Ω_i, and we assume that $v_A \ll c$, condition (13) is always satisfied. Furthermore, we assume that δ is large enough to satisfy conditions (10) and (12), and note that in the case of very small electron density ($\delta \sim 0$) the wave modes for such a system of mobile positive ions in a neutralizing background of stationary negative dust grains would be analogous to the magnetoionic modes of electrons in a neutralizing background of stationary positive ions [13]. The case of small electron density would be less likely to occur in laboratory or astrophysical plasmas, so that we do not consider it in detail here.

We consider the one-dimensional case (being interested in evolution parallel to the external magnetic field) and introduce the right(left) circularly polarized magnetic field components $B_\pm = B_x \pm iB_y$. Therefore from Eqs. (7)–(9) we find

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v_z)}{\partial z} = 0, \tag{14}$$

$$\frac{\partial B_\pm}{\partial t} = -\frac{\partial}{\partial z}\left[\frac{1}{\delta}(v_z B_\pm - v_\pm)\right] \mp \frac{i v_A^2}{\Omega_i}\frac{\partial}{\partial z}\left(\frac{1}{\rho\delta}\frac{\partial B_\pm}{\partial z}\right), \tag{15}$$

$$\frac{dv_\pm}{dt} = \mp i\Omega_i \frac{1-\delta}{\delta}(v_z B_\pm - v_\pm) + \frac{v_A^2}{\rho\delta}\frac{\partial B_\pm}{\partial z} \tag{16}$$

and

$$\frac{dv_z}{dt} = \mp \frac{i\Omega_i}{2}\frac{1-\delta}{\delta}(v_\pm B_\pm^* - v_\pm^* B_\pm) - \frac{v_A^2}{2\rho\delta}\frac{\partial}{\partial z}(|B_\pm|^2), \tag{17}$$

where the right(left) circularly polarized velocity component has been introduced analogously to the magnetic field component: $v_\pm = v_x \pm i v_y$. The system of Eqs. (14)–(17) is basic for our further investigation.

III. LINEAR DISPERSION

Linearizing Eqs. (14)–(17) with respect to the wave magnetic field, we find

$$\left(\frac{\partial^2}{\partial t^2} \mp i\Omega_i \frac{1-\delta_0}{\delta_0}\frac{\partial}{\partial t} - \frac{v_A^2}{\delta_0}\frac{\partial^2}{\partial z^2} \pm i\frac{v_A^2}{\Omega_i \delta_0}\frac{\partial^3}{\partial t \partial z^2}\right) B_\pm = 0, \tag{18}$$

where $\delta_0 = n_{e0}/n_{i0}$ measures the unperturbed electron and ion number density imbalance. Thus the positive linear frequency of the $\pm$ mode with the wave number $k = k_z$ is given by

$$\omega_{\pm 1} = \mp \frac{\Omega_i}{2\delta_0}\left(1 - \delta_0 + \frac{k^2 v_A^2}{\Omega_i^2}\right) + \left[\frac{\Omega_i^2}{4\delta_0^2}\left(1 - \delta_0 + \frac{k^2 v_A^2}{\Omega_i^2}\right)^2 + \frac{k^2 v_A^2}{\delta_0}\right]^{1/2}. \tag{19}$$

Figure 1 shows the two solutions of the dispersion relation (19) in the form of a plot of the normalized frequency ω/Ω_i against the normalized wave number kv_A/Ω_i for $\delta_0 = 0.8$ and, for comparison, the dust-free case $\delta_0 = 1$. We can divide the range of kv_A up into two regions A and B as shown in Fig. 1. First, we consider the case

$$kv_A \ll \Omega_i. \tag{20}$$

If in addition to condition (20) we have

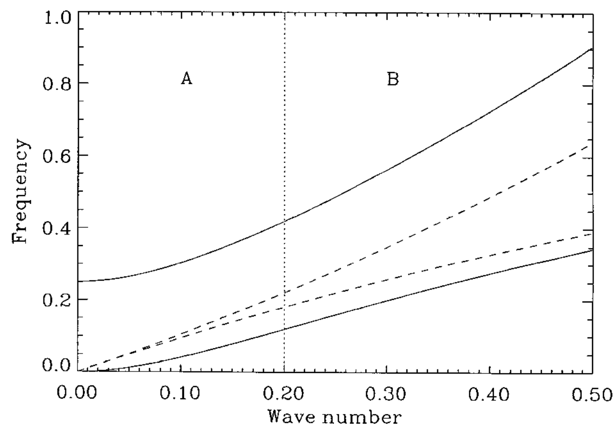

FIG. 1. The normalized frequencies ω/Ω_i of the two modes plotted against normalized wave number kv_A/Ω_i, for $\delta_0 = 0.8$ (solid curves) and $\delta_0 = 1$ (dashed curves). The dotted line delineates the wave number regions A and B.

$$kv_A \ll \Omega_i \frac{1-\delta_0}{\sqrt{2(1+\delta_0)}}, \tag{21}$$

(region A in Fig. 1) then we find the following approximate solutions for the dispersion relation of the waves:

$$\omega_{-1} \equiv \omega_{HF} \simeq \Omega_m \left[1 + \frac{k^2 v_A^2}{(1-\delta_0)^2 \Omega_i^2}\right] \tag{22}$$

and

$$\omega_{+1} \equiv \omega_{LF} \simeq \frac{k^2 v_A^2}{(1-\delta_0)\Omega_i}, \tag{23}$$

where

$$\Omega_m = \Omega_i \frac{1-\delta_0}{\delta_0}. \tag{24}$$

The cutoff frequency of the higher-frequency (HF) mode at $k=0$ as well as the whistler-type dispersion relation (23) of the lower-frequency (LF) mode was obtained by [14]. Note that using $\omega \sim \Omega_m$ in the requirement $\omega \ll \Omega_e$, for validity of the Eqs. (7)–(9) in the case $\delta_0 \ll 1$, yields the condition (10). Using $\omega \sim \Omega_m$ in the condition (13) yields $\delta_0 \gg v_A^2/c^2$, while condition (12) for $\omega \sim \Omega_m$ and $\delta_0 \ll 1$ gives

$$\delta_o \gg \frac{v_A}{c}. \tag{25}$$

If we consider the higher wave number range defined by

$$\Omega_i \frac{1-\delta_0}{\sqrt{2(1+\delta_0)}} \ll kv_A \ll \Omega_i, \tag{26}$$

(region B in Fig. 1) then we obtain

$$\omega_{HF,LF} \simeq kv_A \left(\frac{1+\delta_0}{2}\right)^{1/2} \pm \frac{\Omega_i}{2}\left(1 - \delta_0 + \frac{k^2 v_A^2}{\Omega_i^2}\right), \tag{27}$$

i.e., an almost linear dispersion relation, with a displacement in frequency due to the charge imbalance and ion-cyclotron dispersion, of opposite sign for the two modes. Note that the region B exists as a distinct region only in the case when the electron density is not too small; we distinguish the region B specifically since for many applications this case is of particular interest. For example, for dusty interstellar clouds whose Alfvén waves are strongly affected by the dust, $(1-\delta)$ is typically of order 10^{-4} [16].

Finally, if we consider waves with

$$\Omega_i \ll k v_A, \tag{28}$$

we have the following dispersion relations for the high-frequency and low-frequency mode, respectively:

$$\omega_{HF} \simeq \frac{\Omega_i}{\delta_0}\left(1+\frac{k^2 v_A^2}{\Omega_i^2}\right) \tag{29}$$

which is the whistler mode, and

$$\omega_{LF} \simeq \Omega_i, \tag{30}$$

which is the usual resonant electromagnetic ion-cyclotron mode.

Note that as a solution of the dispersion equation (18), we also have two negative-frequency modes with frequencies $\omega_{\mp}^2 = -\omega_{\pm}^1$. For the wave fields, the following obvious relationship holds:

$$B_{\mp}^{(1)} = (B_{\pm}^{(2)})^*, \tag{31}$$

where the fields $B_{-}^{(1)}$ as well as $B_{+}^{(2)}$ correspond to the higher-frequency mode, whereas $B_{+}^{(1)}$ and $B_{-}^{(2)}$ correspond to the lower-frequency mode.

We now proceed to analyze the nonlinear behavior of the modes satisfying condition (20).

IV. NONLINEAR CONTRIBUTIONS

The linear electromagnetic modes discussed above do not involve perturbations of ρ or v_z. However it is apparent from the nonlinear equations (14)–(17) that fluctuations in these quantities will provide a nonlinear modification of the electromagnetic modes, just as for the dust-free case. The fluctuations in ρ and v_z are assumed to be of small amplitude [of order $|B_\pm|^2$ from Eq. (17)], so we can linearize Eqs. (14)–(17) in these quantities, to obtain the following equations:

$$\frac{\partial B_\pm}{\partial t} = -\frac{1}{\delta_0}\frac{\partial}{\partial z}\left[v_z B_\pm - \left(1-\frac{1-\delta_0}{\delta_0}\Delta\rho\right)v_\pm\right] \mp \frac{i v_A^2}{\delta_0 \Omega_i}\frac{\partial}{\partial z}\left[\left(1-\frac{\Delta\rho}{\delta_0}\right)\frac{\partial B_\pm}{\partial z}\right], \tag{32}$$

$$\frac{\partial v_\pm}{\partial t} + v_z \frac{\partial v_\pm}{\partial z} = \mp i\Omega_m\left[v_z B_\pm - \left(1-\frac{\Delta\rho}{\delta_0}\right)v_\pm\right] + \frac{v_A^2}{\delta_0}\left(1-\frac{\Delta\rho}{\delta_0}\right)\frac{\partial B_\pm}{\partial z}, \tag{33}$$

$$\frac{\partial \Delta\rho}{\partial t} + \frac{\partial v_z}{\partial z} = 0, \tag{34}$$

and

$$\frac{\partial v_z}{\partial t} = \mp \frac{i\Omega_m}{2}(v_\pm B_\pm^* - v_\pm^* B_\pm) - \frac{v_A^2}{2\delta_0}\frac{\partial}{\partial z}(|B_\pm|^2), \tag{35}$$

where we have used the relations

$$\rho \approx 1+\Delta\rho, \quad \delta \approx \delta_0 + (1-\delta_0)\Delta\rho. \tag{36}$$

Note that for the linearization used in Eqs. (32)–(35) to be valid, the condition $\Delta\rho \ll \delta_0$ must hold. If $\delta_0 \sim 0$, i.e., if the electron density is very small, the resulting nonlinear equations would have a different structure to Eqs. (32)–(35). We will not discuss that case here, since it is a subject for a separate investigation which will be reported elsewhere.

Taking the time derivative of Eq. (32), and using Eq. (33), we obtain the following second order equation for $B_\pm$, linearized in $\Delta\rho$ and v_z:

$$\frac{\partial^2 B_\pm}{\partial t^2} = \pm i\Omega_m \frac{\partial B_\pm}{\partial t} + \frac{v_A^2}{\delta_0}\left(1 \mp i\frac{1}{\Omega_i}\frac{\partial}{\partial t}\right)\frac{\partial}{\partial z}\left[\left(1-\frac{\Delta\rho}{\delta_0}\right)\frac{\partial B_\pm}{\partial z}\right] - \frac{1}{\delta_0}\frac{\partial}{\partial z}\left[\frac{\partial}{\partial t}\left(v_z B_\pm + \frac{1-\delta_0}{\delta_0}v_\pm \Delta\rho\right) + v_z \frac{\partial v_\pm}{\partial z} \pm i\Omega_m v_\pm \Delta\rho\right], \tag{37}$$

which generalizes the linear equation (18) to include the nonlinear terms involving $\Delta\rho$ and v_z. Rearranging linear and nonlinear terms in Eq. (37), we can rewrite it as

$$\frac{\partial}{\partial t}\left[\frac{\partial B_\pm}{\partial t} \mp i\Omega_m B_\pm \pm \frac{i v_A^2}{\delta_0 \Omega_i}\frac{\partial^2 B_\pm}{\partial z^2} + \frac{\partial}{\partial z}\left(\mp \frac{i v_A^2 \Delta\rho}{\delta_0 \Omega_i}\frac{\partial B_\pm}{\partial z} + \frac{v_z B_\pm}{\delta_0} + \frac{v_\pm \Delta\rho}{\delta_0^2}\right)\right] = \frac{1}{\delta_0}\frac{\partial^2}{\partial z^2}(v_A^2 B_\pm - v_z v_\pm). \tag{38}$$

Thus we see that the imbalance between the electron and ion nonlinear currents (due to electron capture by the dust) leads to the amplification of the nonlinear terms (the factor δ_0 in their denominators). We now consider the nonlinear modifications of the linear modes described in Sec. III for the two lowest wave number regions A and B.

A. Region *A*

We consider separately the nonlinear modification of the higher-frequency and lower-frequency linear modes with wave numbers satisfying Eq. (21).

1. Higher-frequency mode

The wave is assumed to propagate at a frequency close to Ω_m, so that we introduce the envelope amplitudes of the wave magnetic field and ion velocity perturbations as

$$B_{-1} = b_1 \exp(-i\Omega_m t), \quad v_{-1} = v_1 \exp(-i\Omega_m t). \tag{39}$$

The nonlinear term in Eq. (37) may be simplified using the approximate linear relations for b_1 and v_1:

$$-i\Omega_m+\frac{\partial b_1}{\partial t}\approx\frac{1}{\delta_0}\frac{\partial v_1}{\partial z}+\frac{iv_A^2}{\delta_0\Omega_i}\frac{\partial^2 b_1}{\partial z^2} \tag{40}$$

and

$$\frac{\partial v_1}{\partial t}\approx\frac{v_A^2}{\delta_0}\frac{\partial b_1}{\partial z}. \tag{41}$$

The dominant contribution to Eq. (38) is then

$$i\frac{\Omega_m}{\delta_0}\left[(1+\delta_0)\frac{\partial}{\partial z}(v_zB_-)-i\Omega_m\Delta\rho B_-\right], \tag{42}$$

and the result of using the envelope expressions (39) and retaining the largest dispersion and nonlinear terms, is the following equation for the envelope amplitude of the magnetic field:

$$i\frac{\partial b_1}{\partial t}+\frac{v_A^2}{\delta_0^2\Omega_m}\frac{\partial^2 b_1}{\partial z^2}+i\frac{1+\delta_0}{\delta_0}\frac{\partial(v_zb_1)}{\partial z}+\Omega_m\Delta\rho b_1=0. \tag{43}$$

Thus we see that the charge imbalance leads to the appearance of an additional nonlinear term containing the variation of the ion density $\Delta\rho=\Delta n_i/n_{i0}$; the corresponding term is absent in the dust-free case.

For the parallel component of the ion velocity perturbation, we have from Eq. (35)

$$\frac{\partial v_z}{\partial t}=-\frac{i\Omega_m}{2}(v_1^*b_1-v_1b_1^*)-\frac{v_A^2}{2\delta_0}\frac{\partial(|b_1|^2)}{\partial z}. \tag{44}$$

Equations (43) and (44), together with the linearized continuity equation (34), form the set of equations describing evolution of the higher frequency mode amplitude.

2. Lower-frequency mode

In this case, we have the approximate linear relations for $b_2=B_+$ and $v_2=v_+$

$$\frac{\partial b_2}{\partial t}\approx\frac{1}{\delta_0}\frac{\partial v_2}{\partial z}-\frac{iv_A^2}{\delta_0\Omega_i}\frac{\partial^2 b_2}{\partial z^2} \tag{45}$$

and

$$\frac{\partial v_2}{\partial t}\approx i\Omega_m v_2+\frac{v_A^2}{\delta_0}\frac{\partial b_2}{\partial z}. \tag{46}$$

Retaining the dominant dispersive and nonlinear terms in Eq. (38), we obtain

$$i\frac{\partial b_2}{\partial t}+\frac{v_A^2}{\delta_0\Omega_m}\frac{\partial^2 b_2}{\partial z^2}-\frac{1}{\delta_0\Omega_m}\frac{\partial}{\partial z}\left[\frac{\partial}{\partial t}\left(v_zb_2+\frac{v_2\Delta\rho}{\delta_0}\right)+\frac{\partial}{\partial z}(v_zv_2)\right]=0. \tag{47}$$

Using the linear equations (45) and (46), and ignoring nonlinear terms containing higher order derivatives (we note that even in the linear approximation we do not take into account time derivatives of order higher than unity, and space derivatives of order higher than two), we find

$$i\frac{\partial b_2}{\partial t}+\frac{v_A^2}{\delta_0\Omega_m}\frac{\partial^2 b_2}{\partial z^2}-\frac{1}{\delta_0\Omega_m}\frac{\partial}{\partial z}\left(b_2\frac{\partial v_z}{\partial t}\right)=0. \tag{48}$$

An equation for v_z can also obtained using the linear relations (45) and (46)

$$\frac{\partial v_z}{\partial t}=-\frac{1}{2}\left(b_2^*\frac{\partial v_2}{\partial t}+b_2\frac{\partial v_2^*}{\partial t}\right)\approx\frac{v_A^4}{2(1-\delta_0)^2\Omega_i^2}\frac{\partial}{\partial z}\left(\frac{\partial^2}{\partial z^2}\left|b_2\right|^2-3\left|\frac{\partial b_2}{\partial z}\right|^2\right). \tag{49}$$

Thus we can conclude that the nonlinear contribution to Eq. (48) in the approximation used is negligible since it contains high orders of field derivatives. We also should mention that for the lower-frequency mode, the dust dynamics can be of large importance [17], so that Eqs. (48) and (49) should be modified to include dust dynamic terms. Since we are only considering immobile dust grains in this paper, we do not study the nonlinear properties of the lower-frequency mode further.

B. Region *B*

For this range of wave numbers, defined by Eq. (26), the ion-cyclotron and dust dispersive terms in the dispersion relation Eq. (27) are similar size corrections to the usual Alfvén wave dispersion relation. The same method of stretched coordinates as used for the dust-free case [1] with an expansion parameter ϵ, where

$$\frac{kv_A}{\Omega_i}=\mathcal{O}(\epsilon)\quad\text{and}\quad\frac{\Omega_m}{kv_A}=\mathcal{O}(\epsilon), \tag{50}$$

can be used to simplify Eq. (37) or Eq. (38), with the resulting derivative nonlinear Schrödinger equation (DNLS)

$$\frac{\partial\widetilde{B}_\pm}{\partial t}+v_A\frac{\partial\widetilde{B}_\pm}{\partial z}+\frac{v_A^2}{2\delta_0^2}\left(1-\frac{1}{2\delta_0}\right)\frac{\partial}{\partial z}(|\widetilde{B}_\pm|^2\widetilde{B}_\pm)\pm i\frac{v_A^2}{2\delta_0\Omega_i}\frac{\partial^2\widetilde{B}_\pm}{\partial z^2}=0, \tag{51}$$

where

$$B_\pm=\widetilde{B}_\pm\exp\left(\pm i\frac{\Omega_m}{2}t\right). \tag{52}$$

We note that the conditions (50) imply that $\delta_0=1-\mathcal{O}(\epsilon^2)$, so that to dominant order in ϵ the coefficients of the z derivatives in Eq. (51) are not affected by the dust. Also the dust term in Eq. (35) contributing to the slow variation of v_z is much smaller than the other term on the right hand side of Eq. (35). Thus the main effect of the dust in this case is a frequency shift given by Eq. (52). Note that Eq. (51) holds under condition (26), in particular, this means that the specific case $\delta\approx 1/2$ when the nonlinear term of Eq. (51) vanishes, is excluded.

V. NONLINEAR WAVES

We now seek wave solutions of the weakly nonlinear equations derived in the preceding section for the higher-frequency mode of region A. Note that the dispersion relation (22) obtained under assumption (21) has no analogies in the dust-free case, therefore the results in this section are valid only when Eq. (21) is satisfied and thus cannot be related to the nonlinear waves studied in [1–5] which correspond to the limit $\delta_0 \to 1$ under condition (26), see also Sec. IV B.

We start with Eqs. (43) and (44) describing the nonlinear evolution of the envelope amplitude of the magnetic field for this mode. First, we have the linear solution of the equations in the form

$$b_1 = b_{10} \exp(-i\Omega t + iKz), \quad v_1 = v_{10} \exp(-i\Omega t + iKz), \tag{53}$$

where b_{10} and v_{10} are constants, and

$$\Omega \simeq \frac{K^2 v_A^2}{\delta_0^2 \Omega_m} \tag{54}$$

is the dispersion correction to the frequency Ω_m.

Now we allow for a slow dependence of the amplitudes b_{10} and v_{10}, as well as their phase Θ, on t and z

$$b_1 = b_{10}(t,z) \exp[i\Theta(t,z)]. \tag{55}$$

We have

$$\Omega(t,z) = -\frac{\partial \Theta}{\partial t}, \quad K(t,z) = \frac{\partial \Theta}{\partial z}. \tag{56}$$

Furthermore, we consider a propagating solution only, where all functions (including phases) depend on

$$Z = z - v_0 t. \tag{57}$$

In this case, for substitution in Eq. (44) we have from Eq. (41),

$$v_1(Z) = -\delta_0 v_0 b_1(Z). \tag{58}$$

Therefore we find that terms containing the velocity v_1 are canceled in Eq. (44) describing the evolution of v_z. This is an important simplification which holds only under the assumption of the dependence of both amplitudes and phase on Z given by Eq. (57). In contrast, the most general solution of the DNLS equation [e.g., Eq. (51) describing waves in region B] assumes different dependences for the amplitudes and the phase, i.e., different carrier and envelope speeds [3].

Assuming that all perturbations vanish at $Z \to \pm\infty$, we find from Eq. (34) and Eq. (44) for the ion density perturbation and parallel component of ion velocity

$$\Delta\rho(Z) = \frac{v_A^2}{2 v_0^2 \delta_0} |b_1(Z)|^2 \tag{59}$$

and

$$v_z(Z) = \frac{v_A^2}{2 v_0 \delta_0} |b_1(Z)|^2. \tag{60}$$

Using Eq. (57), we can write Eq. (43) in the form of a mixed DNLS-NLS equation

$$-i v_0 \frac{db_1}{dZ} + D \frac{d^2 b_1}{dZ^2} + i\mu \frac{d}{dZ}(|b_1|^2 b_1) + \nu |b_1|^2 b_1 = 0, \tag{61}$$

where the coefficients are

$$D = \frac{v_A^2}{\delta_0^2 \Omega_m}, \quad \mu = \frac{(1+\delta_0) v_A^2}{2\delta_0 v_0}, \quad \nu = \frac{\Omega_m v_A^2}{2\delta_0^2 v_0^2}. \tag{62}$$

Furthermore, substituting Eq. (55) into Eq. (61) and separating the imaginary part we find

$$K(Z) = \frac{d\Theta(Z)}{dZ} = \frac{v_0}{2D} - \frac{3\mu b_{10}^2}{4D}, \quad \Omega(Z) = K(Z) v_0. \tag{63}$$

The equation for the real part of Eq. (61) can be written as

$$\frac{d^2 b_{10}}{dZ^2} = -\frac{\partial U(b_{10})}{\partial b_{10}}, \tag{64}$$

where the nonlinear potential is

$$U(b) = \frac{v_0^2}{8D^2} b^2 + \left(\frac{\nu}{4D} - \frac{\mu v_0}{8D^2} \right) b^4 + \frac{\mu^2}{32 D^2} b^6. \tag{65}$$

The potential (65) describes nonlinear cnoidal waves; the corresponding solutions can be written in terms of elliptic functions. Note that when

$$\nu < \frac{\mu v_0}{2D} \quad \text{or } v_A^2 < (1+\delta_0) \delta_0^2 v_0^2, \tag{66}$$

a metastable equilibrium state with $b_{10} \neq 0$ appears. A separatrix dividing the quasiperiodic oscillations in this case corresponds to a localized wave packet. Also, we find that no solutions in the form of standard soliton solutions occur if the conditions opposite to Eq. (66) are applicable.

VI. CONCLUSION

To conclude, we have demonstrated that the presence of a third charged immobile impurity component, such as dust, in a magnetized plasma leads to qualitatively new features in the linear and nonlinear behavior of electromagnetic waves for the frequencies of the order or less than the ion-cyclotron frequency. First, there is a frequency split in the Alfvén wave dispersion relation at small wave numbers due to the change in the linear dispersion (see Fig. 1). Second, the nonlinear properties of the higher-frequency mode are changed compared with impurity-free plasmas; the basic nonlinear equation (43) for the magnetic field envelope contains additional nonlinear term proportional to the charge imbalance and variation of the ion density. We have shown that the nonlinear set of equations (43) and (44), together with the linearized continuity equation (34), in the case of equal carrier and envelope speeds can be reduced to the mixed DNLS-NLS equation which possesses soliton solutions. Further investigations are necessary in the case when the solutions cannot be represented in the form of a constant profile traveling wave. We have also shown that there are no considerable nonlinear effects on the dynamics of the lower-frequency mode for the case of immobile dust grains; in this case, the dust dynamics would be a dominant influence on the properties of the mode.

Finally, we note that the dust-charge fluctuations as well as the grain dynamics, neglected in the present paper, can also affect the nonlinear coupling of the higher-frequency mode. However, these effects are of most importance for waves whose frequencies are much less than the ion-cyclotron frequency. As we have mentioned in Sec. IV, this is also the reason why we do not consider in the present paper (within the approximation of immmobile dust with a fixed charge) the nonlinear properties of the lower-frequency mode. The investigations [18–21] have addressed some specific features of low-frequency electromagnetic modes due to the dust dynamics and charging. The approximation used in the present paper assumes that all the characteristic frequencies of the nonlinear process are higher than the dust cyclotron frequency and the charging frequency. Note that if these assumptions are violated for averaged motions, the nonlinear coupling will be strongly affected by these effects; in the latter case an analysis similar to that presented in [19] would be necessary. We also mention that a coupling between the Alfvén wave and the low-frequency dust-acoustic wave can occur; the speed of the latter wave is rather small and determined by the frequency of oscillations of the dust particles as well as by the electron and ion temperatures.

ACKNOWLEDGMENTS

Support for this work has been provided by the Australian Research Council.

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